Stabilized vortices in layered Kerr media

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(Received 25 May 2004; revised manuscript received 20 December 2004; published 29 March 2005)

In this paper, we demonstrate the possibility of stabilizing beams with angular momentum propagating in Kerr media against filamentation and collapse. Very long propagation distances can be achieved by combining the choice of an appropriate layered medium with alternating focusing and defocusing nonlinearities with the presence of an incoherent guiding beam which is itself stabilized in this medium. The applicability of the results to the field of matter waves is also discussed.

DOI: 10.1103/PhysRevE.71.036624 PACS number(s): 42.65.Tg, 05.45.Yv, 03.75.Lm

I. INTRODUCTION

Vortices have been a source of fascination since the works of Empedocles, Aristotle, and Descartes, who tried to explain the formation of the Earth, its gravity, and the dynamics of the solar system as due to primordial cosmic vortices. Many interesting problems related to vortices are open in several fields, such as fluid mechanics, superconductivity, superfluidity, light propagation, Bose-Einstein condensation (BEC), cosmology, biosciences, or solid state physics [1–6].

In wave mechanics, a vortex is a screw phase dislocation, or defect [7], where the amplitude of the field vanishes. The phase around the singularity has an integer number of windings, ℓ, which plays the role of an angular momentum. For fields with nonvanishing boundary conditions, this number is a conserved quantity and governs the interactions between vortices as if they were endowed with electrostatic charges. Thus, ℓ is usually called the *topological charge* of the defect.

In optics there has been a strong interest in the so-called *vortex solitons*, i.e., robust distributions of light of vortex type in which nonlinearity could compensate diffraction leading to stationary propagation. However, in self-focusing Kerr media, a finite-size beam containing a vortex always destabilizes and forms a filamentary structure [8]. This also stands for saturable self-focusing nonlinearities [9]. Vortex solitons have been studied in many other different optical systems (see, e.g., the review [10]) and in most realistic cases they tend to be unstable.

In this paper, we propose to use layered Kerr media, which are self-focusing on average, to obtain stable propagation of vortices up to very long distances. Our ideas are also extended to the field of matter waves.

II. STABILIZED SOLITONS

A. Stabilized Townes solitons

The propagation of a paraxial monochromatic beam in a Kerr medium is modeled by equations of the type (in adimensional units)

$$i\frac{\partial u}{\partial z} = -\frac{1}{2}\Delta u + g(z)|u|^2 u, \tag{1}$$

where $u(x,y,z):\mathbb{R}^2\times\mathbb{R}^+\to\mathbb{C}$ is the slowly varying amplitude of the beam envelope, $\Delta=\partial^2/\partial x^2+\partial^2/\partial y^2$, and $g(z)$ is a periodic function accounting for the (spatially modulated) nonlinearity. It is well known that, if g is constant, there exist stationary solutions of Eq. (1) which are of the form $u(\mathbf{r},z)=\Phi_k(\mathbf{r})e^{i\lambda_k z}$ and satisfy

$$\Delta\Phi_k - 2\lambda_k\Phi_k - 2g|\Phi_k|^2\Phi_k = 0. \tag{2}$$

As it is precisely stated in Ref. [11], when g is negative, for each positive λ_k there exist an infinite number of radially symmetric solutions decaying exponentially at infinity. These solutions are characterized by the number of nodes k they have. The one with zero nodes has also the minimum value of the power $I=\int|\Phi_k|$ between all the possible solutions of Eq. (2). It is called the *ground state* or *Townes soliton* and we will denote it as $\Phi_0(r)$. Moreover, the Townes soliton is *unstable* in the sense that small perturbations of a Townes soliton initial data lead the solution of Eq. (1) to collapse or spread.

It has been proposed that an appropriate modulation of the Kerr coefficient of the optical material along the propagation direction could lead to focusing and expansion of the propagating beam in alternating regions, thus yielding a stabilization of the optical beam on average [12–16]. It has been shown that the structure which arises when the nonlinearity is modulated as described above is a stabilized Townes soliton (STS) [17].

B. Partially stabilized vortices

However, Townes solitons are not the only stationary solutions of Eq. (1) for constant g. There also exist vortex-type solutions of the form $u(\mathbf{r},z)=\Phi_\ell(r)e^{i\ell\theta}e^{i\lambda z}$ which are unstable as well. One could naively expect that the same stabilization mechanism proposed in Ref. [12] could be applied to achieve stabilization of these solutions, i.e., to induce alternative expanding and squeezing cycles of the vortex by a

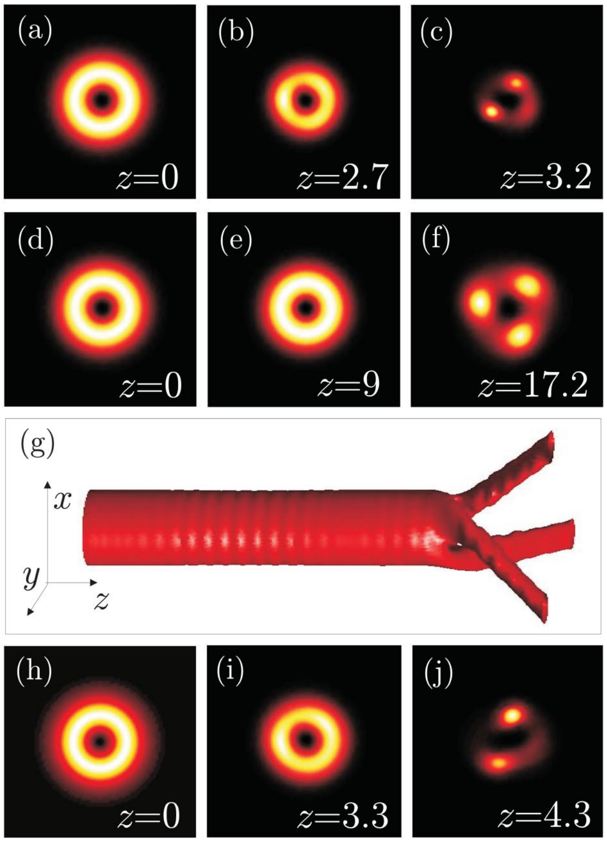

FIG. 1. (Color online) Evolution of initial data of vortex type obtained as a stationary solution of Eq. (1) for $g_v=-24.15$ under different parameter variations. (a)–(c) Evolution for constant $g=-8\pi$ and a noise of 1% in amplitude. (d)–(f) Evolution with modulated nonlinearity $g(z)=-8\pi+20\pi\cos 40z$ without noise. (g) Isosurface plot of $u(x,y,z)$ of the same simulation as in (d)–(f) showing the development of the instability. (h)–(j) Same as (d)–(f) with addition of noise.

periodic modulation of the nonlinear coefficient. To study further this possibility, we have obtained numerically the profiles of vortex solutions of Eq. (1) corresponding to a constant value of $g_v=-24.15$ (the critical one for vortex solutions), by using a standard shooting method. We have then studied the evolution of this stationary solution numerically in different situations using an appropriate pseudospectral method [17]. Obviously if we compute the evolution of this vortex solution for constant $g>g_v$ it would expand, while for $g=g_0<g_v$ it would collapse. In the latter case, the addition of some kind of perturbation breaking the radial symmetry of the initial profile yields the vortex to break up into filaments as can be seen in Figs. 1(a)–1(c), where $g_0=-8\pi<g_v$ and a noise of 1% in amplitude has been added to the initial conditions. In Figs. 1(d)–1(f), we present some snapshots of the evolution of the vortex after the addition of a stabilizing term to the nonlinearity $g(z)=g_0+g_1\cos\Omega z$ for $g_0=-8\pi$, $g_1=20\pi$, $\Omega=40$. We can see how the periodic modulation of the nonlinearity retards the filamentation of the vortex, which now propagates for a longer distance before breaking into several stabilized solitons. Since each emerging beam is close to a STS, the excess energy is eliminated in the form of radiation which is removed by the absorbing boundary conditions of our numerical scheme. However, in Figs. 1(h)–1(j), we see that the addition of noise provokes the early filamentation of the vortex so the stabilization mechanism is not valid any more. In this paper, we have chosen a smooth form for the modulation $g(z)$ but similar results are obtained when $g(z)$ is taken as a piecewise constant function.

We have made an extensive search in the parameter space for modulations of the form $g(z)=g_0+g_1\cos\Omega z$ and have not been able to find any parameter combination allowing stabilization of the vortex. Concerning finite-dimensional reduced models for the evolution of the effective width of the solutions, such as those successfully used for nodeless beams in Refs. [13–16], we must stress that these formulations do not reflect correctly the dynamics and instabilities of vortex solutions.

III. VECTOR SYSTEMS

From the previous analysis it seems that a vortex cannot be stabilized in the framework of Eq. (1), i.e., in scalar systems. Recent works point out the fact that the incoherent interaction of two components could provide, in saturable media, an effective waveguide for the vortex, leading to a more stable behavior [18–20]. Following this idea, we consider now a vector two-component system with Kerr interactions of the form

$$i\frac{\partial u_1}{\partial z}=-\frac{1}{2}\Delta u_1+g(z)(a_{11}|u_1|^2+a_{12}|u_2|^2)u_1, \tag{3a}$$

$$i\frac{\partial u_2}{\partial z}=-\frac{1}{2}\Delta u_2+g(z)(a_{21}|u_1|^2+a_{22}|u_2|^2)u_2, \tag{3b}$$

where $a_{jk}\in\mathbb{R}$ are the nonlinear coupling coefficients and $g(z)$ accounts for the modulation of the nonlinearity. We will denote $I_j=\int_{\mathbb{R}^2}|u_j|^2dxdy$. Although this system is conservative, in our numerical simulations we incorporate absorbing boundary conditions in order to get rid of the radiation. Therefore, in practice, there will be a decrease of I_j during the propagation.

Equations (3) are a two-dimensional extension of the Manakov system [21]. Among other situations, these equations model the propagation of two circularly polarized beams with opposite polarizations (in that case $a_{11}=a_{22}=1$, $a_{12}=a_{21}=2$). In the context of BEC, Eqs. (3) (with an additional trapping term) describe the dynamics of multicomponent quasi-two-dimensional condensates, u_j being the wave functions for the atomic species. The formation of vector solitons composed of appropriate fractions of Townes states has been studied in Ref. [22].

Our idea is to choose $g(z)$ to achieve the stabilization of the Townes soliton u_1. As the coupling terms in Eqs. (3) would provide an effective waveguide for u_2, it seems reasonable that, when $I_2\ll I_1$, the guiding effect will dominate over self-interaction and the vortex could become stabilized.

A. Limit of small u_2

Let us first consider the case of constant g and $I_2\ll I_1$ so that Eqs. (3) become

$$i\frac{\partial u_1}{\partial z} \simeq -\frac{1}{2}\Delta u_1 + g a_{11}|u_1|^2 u_1, \tag{4a}$$

$$i\frac{\partial u_2}{\partial z} \simeq \left(-\frac{1}{2}\Delta + g a_{21}|u_1|^2\right) u_2. \tag{4b}$$

Taking $u_1(r,z)=\Phi_0(r)e^{i\lambda_0 z}$, then Eq. (4b) is a linear two-dimensional Schrödinger problem in which the role of the potential is played by $|\Phi_0|^2$. Following Ref. [23], we can bound the number of ℓ-wave bound states in this potential by

$$N_\ell < (a_{21}g/\ell)\int_{\mathbb{R}^+} r|u_1|^2 dr. \tag{5}$$

Vortex-type solutions with the smallest topological charge are those with $\ell=1$. Thus, taking into account that for a Townes soliton $gI_1=0.931$, we get $N_{\ell=1}<0.931a_{21}$. Therefore, we can expect the existence of a unique $\ell=1$ stationary vortex solution of Eq. (4b) in the case $a_{21}=2$. This will be denoted hereafter as $u_2=V(r)e^{i\theta}e^{i\lambda_v z}$. Let us notice that for $\ell \geqslant 2$ we get always $N_{\ell\geqslant 2}<1$, thus ruling out the possibility of obtaining higher-order vortices. We have numerically found the profile of this vortex solution $V(r)$ by using a standard shooting method.

Choosing an appropriate modulation for g allows us to stabilize u_1. For Eq. (4b), the potential—the stabilized Townes soliton—oscillates with a fast frequency of the order of Ω and another slower one of dynamical origin [16]. In this linear case, we can apply to u_2 the quantum-mechanical theory of fast perturbations [24] to account for the effect of the fast modulation on the vortex. The main result of this theory is that the vortex will remain unaffected by the fast perturbation in the potential provided the modulation period T satisfies $\Delta_{u_2}\overline{H} \ll 1/T$. In our case,

$$\overline{H} = \frac{1}{T}\int_0^T \left[-\frac{1}{2}\Delta + g(z)a_{21}|u_1|^2\right]dz \tag{6}$$

and this inequality imposes $T \ll 1.1$ which requires $\Omega \gg 6$.

On the other side of the spectrum, the STS also has an oscillation of lower frequency, but then adiabaticity allows us to expect a modulation of the same frequency in the vortex oscillations.

To verify these ideas in the limit of small u_2, we have simulated Eqs. (4) with initial conditions $u_1(r,0)=\Phi_0(r)$, $u_2(r,\theta,0)=V(r)e^{i\theta}$, and $g(z)=-2\pi+8\pi\cos 40z$. Full stabilization of the vortex is observed, its oscillations following the pattern predicted above, i.e., there is only a residual fast oscillation in the vortex component and its slow oscillation follows that of the stabilized Townes soliton in u_1.

B. Fully nonlinear regime

Now, we look for solutions of Eqs. (3) with initial data $u_1(r,0)=\Phi_0(r)$, $u_2(r,\theta,0)=\alpha V(r)e^{i\theta}$ and $g(z)=-2\pi+8\pi\cos 40z$ [similar results are found starting with stationary solutions of Eqs. (3)].

For small values of α (e.g., $\alpha=0.1$ as in Fig. 2), the vortex

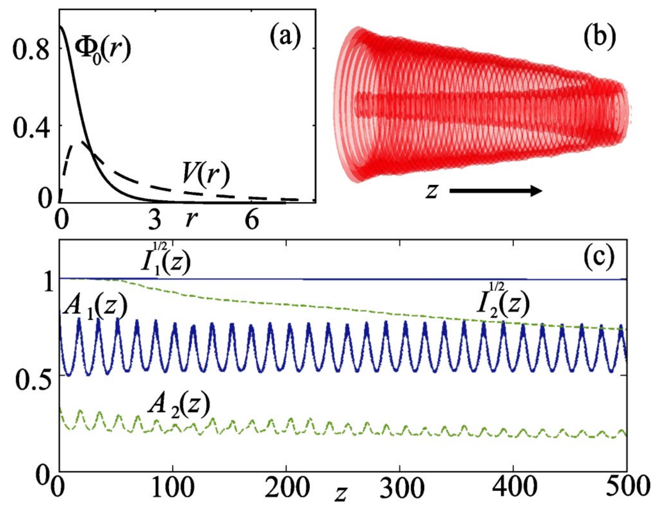

FIG. 2. (Color online) Solutions of Eqs. (3) for initial data $u_1(r,0)=\Phi_0(r)$, $u_2=\alpha V(r)e^{i\theta}$ with $\alpha=0.1$ in a grid of 810×810 points on $(x,y)\in[-40,40]$, showing stable (but dissipative due to the effect of radiation) propagation of the vortex in the range $z\in[0,500]$. (a) Initial radial profiles. (b) Isosurface plot of $u_2(x,y,z)$ spanning all the propagation range. (c) Evolution of the norms $I_1^{1/2}(z)$, $I_2^{1/2}(z)$ and of the amplitudes $A_1(z)=\max_{(x,y)}|u_1|$, $A_2(z)=\max_{(x,y)}|u_2|$ of both components.

is fully stabilized up to the maximum propagation distances studied. However, a continuous loss of energy is observed during propagation. We think that this power damping is due to radiation emitted by the vortex and it is related to the continuous background oscillations of the stabilizing Townes beam.

For larger values of α (e.g., $\alpha=0.32$ as in Fig. 3), the vortex destabilizes in a spiraling form at long propagation distances due to the effect of nonlinear interactions between the guiding Townes soliton and the vortex. This kind of instability is a known phenomenon which also occurs in other situations where several components interact (see, e.g., Ref. [25]). Although the perturbation is not small, the vortex propagates for very long distances of about 300 propagation units (compare this with the results shown in Fig. 1). In the region of stable propagation, the vortex amplitude decays slowly due to the emission of radiation until a stabilized vector soliton is formed. In this process, both components emit radiation to readjust their norms to satisfy the relation $I_1+I_2=I_{Townes}$[22]. From Fig. 3(i) it is clear that the oscillations of the vortex amplitude basically contain only the slower frequency and that our previous arguments apply here. The addition of noise to the initial data of 1% in amplitude triggers the instability faster, but even in that case the vortex propagates for more than 125 adimensional units before the instability sets in. Since vortices destabilize in a spiraling form in order to get large stable evolution times, it is necessary to align precisely both beam centers. Specifically, a beam center shift of around 1% of the guiding beam size would lead to a decrease of the propagation distance to about 50 adimensional units.

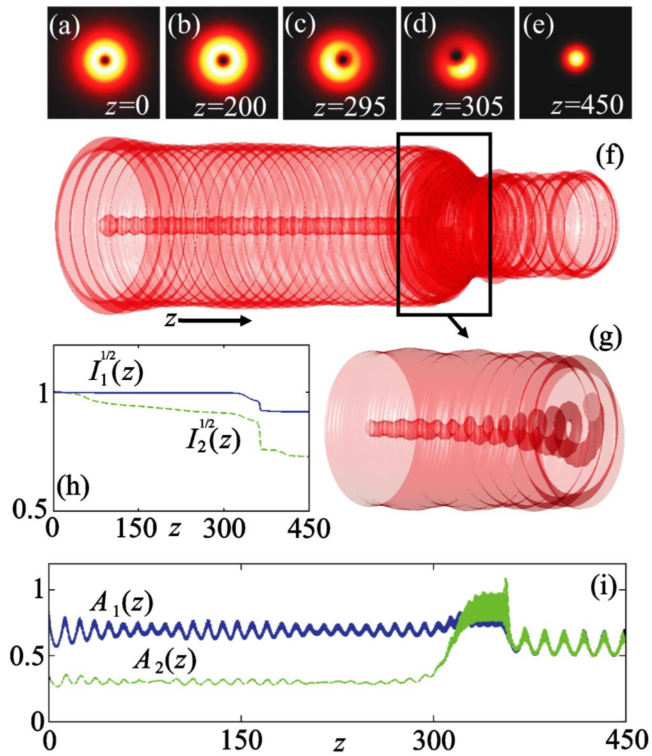

FIG. 3. (Color online) Same as Fig. 2 for $\alpha=0.32$. (a)–(e) Pseudocolor plots of $u_2(x,y,z)$ for $z=0$, 200, 295, 305, 450. (f) Isosurface plot of $u_2(x,y,z)$ spanning all the propagation range. (g) Details of the region in which the vortex spirals out from u_2 for $z\in[280,319]$. (h) Evolution of the norms of both components $I_1^{1/2}(z)$, $I_2^{1/2}(z)$ showing the readjustment of the norms after the vortex is ejected and a stabilized vector soliton is formed. (i) Amplitudes $A_1(z)=\max_{(x,y)}|u_1|$ and $A_2(z)=\max_{(x,y)}|u_2|$ of both components.

IV. PRACTICAL IMPLEMENTATION

A. Layered Kerr media

Equation (1) is obtained from the paraxial propagation equation by rescaling the transverse coordinates with the beam size L and the propagation coordinate with $L^2k_0n_0$, where k_0 is the wave number and n_0 is the linear refraction index. For a Nd:Yag laser with $\lambda=1.064$ μm and a beam size of $L=20\lambda$, we obtain that, for materials with $n_0=1.6$, a propagation distance of $z=1$ in our simulations corresponds to 4 mm approximately. Thus, between 40 (Fig. 1) and 4000 (Fig. 3) layers 335-μm-thick are needed to reproduce our results, which is achievable experimentally. We can also obtain that $n_2|\Psi|^2\simeq 10^{-3}$, where Ψ is the physical beam. As an experimental setup, we consider a periodic structure constructed with solid glass layers of $Ge_{10}As_{10}Se_{80}$ (self-focusing nonlinearity) and a liquid as CS_2 (defocusing thermal nonlinearity) filling the empty spaces between the layers. For $Ge_{10}As_{10}Se_{80}$ we have $n_2=2.2\times10^{-4}$ cm^2/GW for $\lambda=1.064$ μm. Therefore, we need a power $|\Psi|^2\simeq 5$ GW/cm^2, which is quite realistic for a Nd:Yag laser. Finally, by adjusting the length of the pulses, the n_2 value for CS_2 can be suited adequately to reproduce the $g(z)$ function.

A difficulty with a layered setup is the possibility of retroreflection in the interfaces. The reflection coefficient for a planar wave which incides perpendicular to the boundary between two linear optical materials with refractive indexes n_a and n_b is given by

$$R=\left|\frac{n_b-n_a}{n_b+n_a}\right|^2. \tag{7}$$

Taking indexes of the form $n_a=n_0+n_2|\Psi|^2$ and $n_b=n_0+n_2'|\Psi|^2$ (one focusing and the other one defocusing), we can estimate

$$R\approx\left|\frac{(n_2'-n_2)|\Psi|^2}{2n_0}\right|^2. \tag{8}$$

Taking $n_2|\Psi|^2\approx10^{-3}$ and $n_2'=-6/10n_2$ as in our simulations (with $n_0=1.6$ for the two layers), we have $R\approx2.5\times10^{-7}$. Thus, choosing two nonlinear materials with similar linear refractive indexes (as in the case of CS_2 and some Ga:La:S-based glasses), the number of layers that produce a loss of 10% in the input signal would be about 4×10^5, which exceeds the stability range of the vortex and is well below the material losses.

B. Feshbach resonance managed Bose-Einstein condensates

The previous results have also implications in the field of matter waves because of the close analogy of Eqs. (3) with the equations of evolution of a multicomponent Bose-Einstein condensate in the mean-field approximation. The analysis of vortices in dilute-gas BECs has been a very hot topic in recent years, especially after their experimental generation with different setups [4–6].

In multicomponent BEC systems, the interaction coefficients a_{ij} are proportional to the respective scattering lengths. Although Bose-Einstein condensates are fully three-dimensional, the effective two-dimensionality can be achieved by confining the condensate tightly along a specific direction [26,27]. The condition $N_1<0.931a_{21}$ has relevance since it imposes restrictions to the atomic species which can be used to trap a vortex. For instance, the cross-interaction coefficient in multicomponent condensates made of different hyperfine levels of ^{87}Rb does not satisfy this condition. However, bosonic K-Rb mixtures such as the one described in Refs. [28,29] could be used because of the large scattering length of the collisions K-Rb. Our predictions would imply the existence of self-supported vortices which could be generated using Feshbach resonance management techniques.

V. CONCLUSIONS

In this work, we have studied the possibility of stabilizing beams with angular momentum (vortices) propagating in Kerr media against filamentation and collapse. The procedure used consists in taking an appropriate layered medium with alternating focusing and defocusing nonlinearities, so that one can retard the filamentation of the beam. Nevertheless, the addition of a slight noise makes the beam break into filaments very early so the stabilization mechanism is not valid in practical situations. One form of avoiding this and obtaining long propagation distances is to use an incoherent guiding beam previously stabilized which acts as a trapping potential for the vortex. We have also discussed the practical implementation of this stabilizing mechanism in nonlinear optics and Bose-Einstein condensates.

ACKNOWLEDGMENTS

This work has been partially supported by Ministerio de Educación y Ciencia (Spain) under Grants No. BFM2003-02832, No. TIC-2000-1105-C03-01, and by the Junta de Comunidades de Castilla-La Mancha under Grant No. PAC-02-002. G.D.M. is supported by Grant No. AP2001-0535 from Ministerio de Educación, Cultura y Deporte.

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